1 Topological vector spaces and differentiable maps

Definition 1.1 (Topology) *Let S be a set. A subfamily T of $P(S)$, the family of subsets of S, is called a* **topology** *on S iff*

- *The empty set $\emptyset$ is an element of T,*
- *The union of elements of an arbitrary subfamily of T is again in T,*
- *The intersection of elements of a finite set of elements of T is again in T.*

The elements of T are called **open sets**. *A* **basis** *of a topology T is a family B of subsets of S such that every open set is a union of elements of B. A* **subbasis** *of T is a family B of subsets of S such that every open set is a union of finite intersections of elements of B. A topological space is called* **compact** *iff every of its open coverings contains a finite subcovering.*

Definition 1.2 (Metric space) *Let S be a set. A* **metric on** *S is a nonnegative function on $S \times S$ with*

- **Symmetry:** *$d(x, y) = d(y, x)$ for $x, y \in S$,*
- **Faithfulness:** *$d(x, y) = 0 \Rightarrow x = y$,*
- **Triangle inequality:** *$d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in S$.*

A **ball of radius** *r* **around** *p is the set $B_r(p) := \{q \in S | d(p, q) < r\}$.*

The triangle inequality implies the converse of the second relation (faithfulness), thus $d^{-1}(\{0\})$ is the diagonal in $S \times S$. Every metric generates a topology by taking the balls as a basis of a topology.
In the following, let K be $\mathbb{R}$ or $\mathbb{C}$. A vector space $(V, +, \cdot)$ over K is called a topological vector space over K iff $+$ and $\cdot$ are continuous maps. With all the details:

Definition 1.3 *A* **topological vector space (tvs) over** *K is a topological space V together with two continuous maps $+ : V \times V \to V$ and $\cdot : K \times V \to V$ such that $(V, +)$ is an abelian group and with the properties:*

- *If 1 is the neutral element of K then $1 \cdot v = v$ for all $v \in V$*
- *For all $a, b \in K$ and all $v \in V$ holds $a \cdot (b \cdot v) = (ab) \cdot v$*
- *For all $a, b \in K$ and all $v, w \in V$ holds $a \cdot (v + w) = a \cdot v + a \cdot w$ and $(a + b) \cdot v = a \cdot v + b \cdot v$.*

A subset A of a tvs V is called **convex** *iff for $p, q \in A$ and $t \in (0, 1)$ also $tp + (1 - t)q \in A$. Finally, V is called* **locally convex** *iff every neighborhood of a point contains a convex subneighborhood.*

Exercise (1). Show that the inversion $v \mapsto -v$ is continuous in every tvs.
Exercise (2). A metric d on a vector space V is called **translation-invariant** if for all $x, y, z \in V$ we have $d(x+z, y+z) = d(x, y)$. Show that the topology generated by a translation-invariant metric is always compatible with the vector-space structure.

Definition 1.4 *Let V be a vector space over K. A* **seminorm on** *V is a nonnegative function $|| \cdot || : V \to \mathbb{R}$ with (for all $v, w \in V, a \in K$)*

- $||av|| = |a| \cdot ||v||$
- $||v + w|| \leq ||v|| + ||w||$.

This implies that $||0|| = 0$. If $||v|| = 0 \Rightarrow v = 0$, $||\cdot||$ is called **norm**. *A positive definite sesquilinear form on V is called* **scalar product**.

Every norm $||\cdot||$ generates a metric d (and thereby a topology) by $d(p,q) := ||p-q||$. This enables us to introduces subclasses of tvs with additional structures: spaces with scalar product, normed spaces and metric vector spaces. If only the topological structure of these spaces is considered we speak of scalable, normable and metrizable tvs.

Example 0: Show that $\mathbb{R}^n$ with the usual product topology can be thought of as coming from choosing a basis and using the associated scalar product. Show that this is a tvs **(exercise(3))**.
Example 1: Let S be a set and $B(S,K)$ the family of bounded maps from S to K. Then the map $n : B(S) \to \mathbb{R}$ given by $n(f) := \sup_S |f|$ is a norm: $\sup_S(|\lambda f|) = |\lambda| \sup_S |f|$ and $\sup_S |f + g| \leq \sup_S |f| + \sup_S |g|$, and also $||v|| = 0$ implies $v = 0$. Note that here we get a tvs, even a normed space, without any topology on S.
Exercise (4): How can you construct the space $B(S, K^n)$ of bounded maps from S to K^n out of $B(S,K)$?

In particular, we can give the space of bounded real sequences the structure of a normed vector space. If we want to generalize this to spaces of arbitrary real sequences $\mathbb{R}^{\mathbb{N}}$, we see that we cannot find easily a norm on this space (later we will show that there is none), but we *can* find a metric. To this purpose, we now consider a technical tool for defining metrics. Let, for every $i \in \mathbb{N}$, $\phi_i : [0,\infty) \to [0,\infty)$ a continuous, strictly increasing, concave function with $\phi_i(0) = 0$. Define $\Phi_i := \sup \phi_i = \lim_{t\to\infty} \phi_i(t)$. If there is a $C \in \mathbb{R}$ with $\Phi_n < C$ for all but finitely many n, then the sequence of functions is called **nice** (or **essentially bounded**). If $\sum_{n\in\mathbb{N}} \Phi_n < \infty$ we will call the sequence of functions **supernice** (or **summable**). In particular, every element of a supernice sequence is a bounded function.
Exercise (5): Show that ϕ given by $\phi_i(t) := 2^{-i} \cdot t/(t+1)$ is a supernice sequence of functions.

Example 2: Consider the space $\mathbb{R}^{\mathbb{N}}$ of real sequences, with the compact-open topology. The latter, made for topologize spaces of maps between topological spaces A and B, is the topology generated by the subbasis consisting of the sets $(K,O) := \{f \in B^A | f(K) \subset O\}$ where K is running through the compact sets in A and O is running through the open sets in B.
Exercise (6): Show that for any ϕ a supernice sequence of functions, the metric $D_\phi(s^1, s^2) = \sum_{i\in\mathbb{N}} \phi_i(|s_i^1 - s_i^2|)$ defines on $\mathbb{R}^{\mathbb{N}}$ a compatible metric and thus a tvs structure. Equally, every nice sequence of functions ψ, via the metric $d_\psi(s_1, s_2) = \sup_{i\in\mathbb{N}} \psi_i(|s_i^1 - s_i^2|)$, determines a compatible metric and thus a tvs structure on $\mathbb{R}^{\mathbb{N}}$.
Remark. We will see after some pages that all these metrics give rise to the *same* tvs structure.

Example 2a: Let T be a topological space, then consider $C_b^0(T,K) \subset C^0(T,K) \subset K^T$, the space of bounded continuous functions, with the compact-open topology.
Exercise (7): Prove that $f \mapsto \sup_T |f|$ is a norm generating the a topology finer or equal to the compact-open topology with equality if and only if T is compact.

It is clear that if T is compact then $C_b^0(T,K) = C^0(T,K)$ and that in general this is not the case: there are continuous unbounded functions on $\mathbb{R}^n$.

Example 2b: Now, let a general topological space T be given with a compact exhaustion $C_i \subset C_{i+1} \to T$ of T (for $T = \mathbb{R}^n$, what could You choose?). Then, for every supernice sequence ϕ resp. nice sequence ψ, we define the metrics

$$d_\psi(f,g) := d_\psi(i \mapsto \sup_{C_i} |f|) = \sup_{i=1}^{\infty} \psi_i(\sup_{C_i} |f|),$$

$$D_\phi(f,g) := D_\phi(i \mapsto \sup_{C_i} |f|) = \sum_{i=1}^{\infty} \phi_i(\sup_{C_i} |f|).$$

Exercise (8): Show that this metric *does* generate the CO topology, even if T is noncompact!

Example 2c: Let U be an bounded open subset of $\mathbb{R}^n$, then we define $C^k(\overline{U}, \mathbb{R})$ as the set of k times differentiable real functions on U such that each of its multiderivatives has a continuous extension to $\overline{U}$. Given a function f, one considers the set of suprema (C^0-norms) of the multiindexed derivatives

$$\frac{\partial^{|l|} f}{\partial x^l} := \frac{\partial^{|l|} f}{(\partial x_{l(1)})^{i(1)} ... (\partial x_{l(k)})^{i(k)}},$$

where the multiindices are defined the way that $l(j) < l(j+1)$ (by the symmetry of higher derivatives) and, by definition, $|l| = \sum i(j)$. Observe that the finiteness of the single terms is guaranteed by the condition that the function has a C^k extension to the boundary and that $\overline{U}$ is compact. Then we add the terms so obtained, so we get

$$||f||_{C^k(\overline{U})} := \sum_{\text{multiindices } l} ||\frac{\partial^{|l|} f}{\partial x_l}||_{C^0(\overline{U})}.$$

Example 3a: If we consider the space F of smooth functions on U which have a smooth extension to $\overline{U}$, we just consider the sequence $C(f)$ of C^k norms of f and define, for a nice sequence ψ resp. a supernice sequence ϕ, the metrics

$$d_\psi(f-g) := d_\psi(C(f-g)) = \sup_{n \in \mathbb{N}} \psi_n(||f-g||_{C^n(\overline{U})}),$$

$$D_\phi(f-g) := D_\phi(C(f-g)) = \sum_{n \in \mathbb{N}} \phi_n(||f-g||_{C^n(\overline{U})}).$$

Both lead to the same tvs structure: as these topologies are metrizable, they are characterized completely in terms of convergence of sequences. **Exercise(9):** Show that a sequence of smooth functions converges in the topologies defined by these metrics if and only if the sequence converges in every C^k topology. Show that they are metrics which do not come from norms.

Remark: Of course one could consider the C^k-norms on F which might seem better at first sight, but it turns out that these cannot produce the topology one wants to give the space; this will be clearer later on when we consider the concept of completeness.

Example 3b: For C^k or smooth functions on a region U which do not have an extension a priori (e.g. on $\mathbb{R}^n$ we cannot even define easily what an extension should mean) we can do the same thing by choosing a compact exhaustion $C_i \subset C_{i+1} \to U$ and get a two-dimensional infinite table of entries $||f-g||_{C^k(U_i)}$ parametrized by $(k,i) \in \mathbb{N} \times \mathbb{N}$. Then by a counting of $\mathbb{N} \times \mathbb{N}$ we can proceed analogously to the previous example and get metrics defining a tvs structure.

Example 3c: Finally, there are examples of spaces of analytic resp. holomorphic functions $C^\omega_b(B_1(0))$ of bounded analytic functions on $B_1(0)$, $C^\omega(\overline{B}_1(0))$ of extendible analytic functions on $B_1(0)$, $C^\omega(\mathbb{R}^n)$ (with the norm being the evaluation in $B_1(0)$). We will consider these spaces later in the text.

Exercise (10): Show that in a tvs V, for every open neighborhood U of 0 there is an open neighborhood W of 0 with $W+W \subset U$. A subset A in V is called **starshaped** if $\lambda A \subset A$ for all $\lambda \in K$ with $|\lambda| < 1$. Show that V is locally starshaped, i.e., that every neighborhood of 0 contains an open starshaped neighborhood of 0.

In the following, for two tvs A and B, let $L(A, B)$ denote the set of all linear maps from A to B, and $CL(A, B)$ the set of all continuous linear maps from A to B.

Theorem 1.5 *Let V be a tvs, $a \in L(V, K)$. Then the following statements are equivalent:*

1. *a is continuous,*
2. *$ker(a)$ is closed,*
3. *There is a neighborhood U of 0 with $a(U)$ bounded in $\mathbb{R}$.*

Proof. $1 \Rightarrow 2$ is obvious and $3 \Rightarrow 1$ an easy **exercise (11)**. So let us show $2 \Rightarrow 3$. Assume that a has a closed kernel. Fix $v \in V \setminus ker(a)$. Then there is an open neighborhood $\tilde{U} = v + U_0$ of v disjoint from $ker(a)$. Choose a starshaped $U \subset U_0$. Then, for every $u \in U$ we have $a(u) \neq \pm a(v)$ (*). The image $a(U)$ is starshaped in K, too, and by (*) contained in the $a(v)$-disk in K. □

In topological vector spaces there is a definition of sequences whose terms 'asymptotically stay arbitrarily close together':

Definition 1.6 *Let V be a tvs. A sequence x_n in V is called* **vector-Cauchy** *iff for every neighborhood U of 0 there is an $n \in \mathbb{N}$ with:*

$$b, c \geq n \Rightarrow x_b - x_c \in U.$$

On a metric space we can make an analogous definition:

Definition 1.7 *Let (M, d) be a metric space. A sequence x_n in M is called* **metric-Cauchy** *iff for every $\epsilon > 0$ there is an $n \in \mathbb{N}$ with*

$$b, c \geq n \Rightarrow d(x_b, x_c) < \epsilon.$$

The natural question staring at the last two definitions is: If we have a tvs whose topology is generated by a metric, do the two of the preceeding definitions coincide? The following theorem tells us that they do if the metric is **translational-invariant**, i.e. if for all $v, w, x \in V$ we have $d(v + x, w + x) = d(v, w)$:

Theorem 1.8 *Let V be a tvs with a translational-invariant metric generating the topology. Then a sequence in V is vector-Cauchy if and only if it is metric-Cauchy.*

The **proof** is an easy **exercise (12)**. □

Now there is another important connection between convergence and the Cauchy property:

Theorem 1.9 *A convergent sequence in a tvs is vector-Cauchy.*

Proof. Similar to the previous proof **(exercise (13)!)**

It is quite reasonable to distinguish those tvs in which also the converse of the previous theorem holds:

Definition 1.10 *A tvs V is called* **(sequentially) complete** *iff every vector-Cauchy sequence converges.*

Remark: In non-metrizable tvs there is a stronger notion of completeness which we will meet later.

Definition 1.11 *A* **Fréchetable space** *is a complete metrizable locally convex tvs. A* **Fréchet space** *is a locally convex space together with a compatible* translation-invariant *complete metric. A* **Banachable space** *is a tvs which has a compatible complete norm. A* **Banach space** *is a pair consisting of a Banachable space and a compatible complete norm. A* **Hilbertable space** *is a tvs which has a compatible complete scalar product. A* **Hilbert space** *is a pair consisting of a Hilbertable space and a compatible complete scalar product.*

Remark-Warning: In the literature, what we call *Fréchetable* is often called *Fréchet* and what we call *Fréchet* is often called *metric Fréchet*.
Exercise (14): Show that any Banach space is locally convex (thus, any Hilbert space is a Banach space and any Banach space is a Frechet space).

To justify further the first part of the definition, let us give the following theorem:

Theorem 1.12 *Every Frechet space has a compatible translation-invariant metric (which then is automatically complete).*

Proof. Let δ be a compatible metric for the Fréchetable space, then define a new metric d by $d(x, y) := \delta(x - y, 0)$. This metric is obviously translation-invariant and it is compatible as well. To show this, one has to prove that every open set contains a ball and vice versa. Thus, given an open set U, choose a point $p \in U$ and consider the set $U - p$. Because the vector addition is continuous, this is an open neighborhood of 0 and contains therefore a ball $B^d_\epsilon(0) = B^\delta_\epsilon(0)$. Then U contains the ball $B^d_\epsilon(p)$. Conversely, given a d-ball $B^d_r(p)$, we want to show that it is open. Because of continuity of vector addition, it is enough to show that $B^d_r(p) - p = B^d_r(0)$ is open. But this coincides with $B^\delta(0)$ which is open by definition. $\square$

Theorem 1.13 *A subset A of a Fréchet space F with the induced metric is complete if and only if it is closed in F.*

Proof: Exercise (15). $\square$

Example 4a: In $B(\mathbb{N}, K)$ consider the following subspaces: the space of finite sequences $B_c(\mathbb{N}, K) := \{a \in B(\mathbb{N}, K) | \exists N \in \mathbb{N} : a_n = 0 \forall n > N\}$, the space of sequences converging to zero $B_0(\mathbb{N}, K) := \{a \in B(\mathbb{N}, K) | \lim_{n\to\infty} a_n = 0\}$ and the space $B_l(\mathbb{N}, K) := \{a \in B(\mathbb{N}, K) | \exists k \in K : \lim_{n\to\infty} a_n = k\}$, all with the restriction of the norm of $B(\mathbb{N}, K)$.
Exercise (16): Show that $B_c(\mathbb{N}, K) \subset B_0(\mathbb{N}, K) \subset B_l(\mathbb{N}, K)$. A subspace A of a vector space V is said to be **complemented** if there is a second subspace B with $V = A \oplus B$. If the complement has finite dimension n one sais that A **has codimension** n. What is the codimension of $B_0(\mathbb{N}, K)$ in $B_l(\mathbb{N}, K)$? Show that $B_c(\mathbb{N}, K)$ is not closed in $B_0(\mathbb{N}, K)$.

Theorem 1.14 *Both $B_0(\mathbb{N}, K)$ and $B_l(\mathbb{N}, K)$ are closed in $B(\mathbb{N}, K)$.*

Example 4b: The concepts from Example 4a generalize to topological spaces T instead of $\mathbb{N}$, and to metric vector spaces V instead of K: $B(T, V)$ with the norm $||f|| := \sup\{d(f(t), 0)|t \in T\}$, as well as its subspaces (for $a \in V$)

$B_l(T, V) := \{f \in B(T, V)|\exists v \in V : f^{-1}(V \setminus B_\epsilon(v)) \text{ is compact}\}$

The space $B_0(T, V)$ corresponds of course to the case $v = 0$. In the case of metric spaces M instead of T there is another possible generalization:

$B_d(T, V) := \{f \in B(T, V)|\exists v \in V : f^{-1}(V \backslash B_\epsilon(v)) \text{ is bounded (contained in a ball)}\}$

We will see when these spaces differ one from the other. Now the last space is $B_c(T, V)$, the space of functions with compact support.

Exercise (17). A topological space A is called **σ-compact** iff it has a countable compact exhaustion $C_n \subset C_{n+1} \to A$. Show that if T contains a noncompact σ-compact subset, $B_c(T, V)$ is not closed in $B(T, V)$, and that, in the case of T being a metric space containing a non-compact σ-compact subset, $C_c(T, V)$ is not closed in $C_b(T, V)$ (You might want to begin with $T = \mathbb{R}$, the idea is always the same).

Exercise (18): Show that the $C^k(\overline{U})$ spaces defined above are complete (it is a typical 3ϵ-argument You can find in any analysis textbook - if You want to spoil Yourself the fun to discover it on Your own...).

Let $B_m(\mathbb{N}, V) \subset V^{\mathbb{N}}$ (with the topology of pointwise convergence, which coincides with the CO topology in this case as every compact set is finite) be the space of vector-Cauchy sequences in a topological vector space V. Obviously, it contains $B_0(\mathbb{N}, V)$. The idea behind the construction of the following space, the completion of V, consists of including all possible limits of sequences in V. Those who feel unconfortable with the definition should first read the second remark after it.

Definition 1.15 (Completion) *Let V be a tvs. Then the space $\overline{V} := B_m(\mathbb{N}, V)/B_0(\mathbb{N}, V)$ with the quotient topology is called the* **completion** *of V.*

Theorem 1.16 *For every tvs V, its completion $\overline{V}$ is complete. There is a natural linear embedding of V in $\overline{V}$. If $V \subset W$ was equipped with the subspace topology and W is a complete tvs that then $\overline{V}$ is the topological closure of V in W (so that using the same symbol for both is justified). Thus, the completion of a complete tvs V is V itself.*

Proof: Exercise (19). □

Remark: The last property of the preceding theorem is a universal property which characterizes the completion $\overline{V}$ as the smallest complete tvs containing V: If there is a complete tvs containing V then it contains $\overline{V}$ a well.

Remark: At first sight, it seems strange that in a definition of a completion of a space V, a space of sequences in V appears, as the latter is much larger. But we divide out a space which is also very large. Observe that in the case of a complete space every Cauchy sequence converges, and then between convergent sequences and sequences converging to 0 the difference is not so large anymore...

Theorem 1.17 *If V carries a compatible metric resp. norm resp. scalar product, we can construct a compatible metric resp. norm resp. scalar product on $\overline{V}$ (which then is automatically complete).*

Proof. Let us begin with the case of a metric. Given two elements $[v], [w]$ of the completion $\overline{V}$, we choose two representatives v, w, i.e. Cauchy sequences in V, and define

$$d^{\overline{V}}([v],[w]) := \lim_{n\to\infty} d^V(v_n, w_n).$$

As v and w are Cauchy sequences, the triangle equality in V implies that $d^V(v_n, w_n)$ is a Cauchy sequence in $\mathbb{R}$, and as $\mathbb{R}$ is complete, the limit exists. For the other cases proceed analogously and set

$$||[v]||^{\overline{V}} := \lim_{n\to\infty} ||v_n||^V$$

$$\langle [v],[w]\rangle^{\overline{V}} := \lim_{n\to\infty} \langle v_n, w_n\rangle^V.$$

This concludes the proof. □

Example 5: Let U be an bounded open subset of $\mathbb{R}^n$, then on $C^k(\overline{U})$, apart from its original norm, we can consider the norm

$$||f||_{k,p} = \sum_{|i|\le k} ||\frac{\partial^{|i|} f}{\partial x^i}||_{L^p(\overline{U})}.$$

where $||g||_{L^p(\overline{U})} := \sqrt[p]{\int_{\overline{U}} |g|^p}$ and the integral is meant with respect to the Lebesgue measure. **Exercise (20):** meditate a while about the question why this is well defined, the Lebesgue measure is a Borel measure, the region is compact, etc. Then find a (k,p)-Cauchy sequence without a limit in C^k showing that C^k with this (k,p)-metric is incomplete!
Now the spaces $W^{k,p}(U)$ are defined as the completion of $(C^k(\overline{U}), ||\cdot||_{k,p})$. In the case of $p=2$ we consider a scalar product instead of a norm.
Remark: This definition is not entirely satisfying as, for example, it is unclear how to interpret the elements of the completed space: are they functions in any sense? Later we will see that there is another possible definition or characterization for $W^{k,p}$.
Example 6: Linear subspaces of those above, C_0^k (functions of compact support) with a subspace topology: not complete in any sense. The space of asymptotic functions

$$C_l^k(U) := \{f \in C^k(U) | \exists g \in C^k(U) \forall \epsilon > 0 \exists K \subset U \text{ compact} : ||f-g||_{C^k(U\setminus K)} \le \epsilon\},$$

or e.g. periodic functions or solutions of linear differential equations.
Example 7: The duals of the spaces above, more generally, the spaces of linear maps between them. Spaces of sequences in infinite-dimensional spaces.
Three examples: $(\mathbb{R}^{\mathbb{N}})^*$, $(C^0(\mathbb{R}))^* = \mu_0(\mathbb{R})$, the space of Borel measures of compact support, and $(C^\infty(\mathbb{R}))^* =: D(\mathbb{R})$, distributions, of which it will turn out that none is metrizable.

Definition 1.18 *Let T be a topological space. A subset A is called* **dense** *if its intersection with every open subset of T is nonempty. It is called* **nowhere dense** *iff the interior of $\overline{A}$ is the empty set. A subset is called* **meager** *iff it is the countable union of nowhere dense sets.*

Remark-Warning: The vast majority of the literature prefers the term *of first category* instead of *meager*.
Exercise (21). Show that the complement of a nowhere dense set is dense.

Obviously this relative property (of a pair of spaces) is preserved by pair homeomorphism, i.e. if H is a homeomorphism between T and U, then A is meager in T

if and only if $H(A)$ is meager in U. Also, with a meager subset A of T, every subset of A is meager as well. Moreover, a countable union of meager subsets is meager.

Theorem 1.19 (Baire's Theorem) *Let M be a complete metric space, let $A : \mathbb{N} \to \tau(M)$ be a sequence of dense open subsets of M. Then their intersection $\bigcap_{i\in\mathbb{N}} A_i$ is dense in M.*

Proof. Let $A \neq \emptyset$ be open in M. Inductively for $n \in \mathbb{N}$ choose, by means of the density assumption, nonempty open balls B_n of radius $1/n$ with $\overline{B}_{n+1} \subset A_{n+1} \cap B_n$. Now set $C := \bigcap_{n\in\mathbb{N}} B_n$. As the midpoints of the balls form a Cauchy sequence, the latter converges to a point in C, thus $C \neq \emptyset$. □

As a corollary, we obtain

Theorem 1.20 *Let M be a complete metric space. Then M is nonmeager in itself.*

Proof. Let a sequence A of nowhere dense subsets be given. Define $C_i := M \setminus A_i$, the sequence of complementa. Then every C_i is dense in M, and Baire's Theorem tells us that their intersection is nonempty, i.o.w. that $M \neq \bigcup_{i\in\mathbb{N}} A_i$. □

Theorem 1.21 *Every finite-dimensional vector space V has exactly one norm topology, which is Banachable.*

Proof. Choose a basis v_i of V and the associated linear map $A : V \to \mathbb{R}^m$. Then pull back the Euclidean norm to V by A. This gives a norm $||\cdot||$, with $||v_i|| = 1$. Assume there is another norm n on V. By using $N := \max_{i=1...m} n(v_i)$, it is easy to see that the identity of V is bounded in both directions. □

Theorem 1.22 *The dual of a Banach space is Banach, too.*

Proof: Exercise (22) □

Theorem 1.23 *Let V be a Hausdorff tvs and $W \subset V$ a linear subspace s.t. W is Fréchetable with its subspace topology. Then W is closed in V.*

Proof. We choose a translation-invariant metric d on W and a point $p \in \overline{W}$. As $B^W_{1/n}(0)$ is open in W, we call an open neighborhoods U of 0 in V **n-fat** iff $U_n \cap W \supset B^W_{1/n}(0)$. Furthermore, for every neighborhood U of 0 we can find a point $q_U \in (p+U)\cap W$. So if we exhaust the family of neighborhoods of 0 by n-fatness (A_n is the class of n-fat neighborhoods), then we have $W \cap U_n := W \cap \bigcap_{A\in A_n} A = B_r(0)$ as there is an open neighborhood A of 0 with $A \cap W = B_{1/n}(0)$. Therefore if we choose points $q_n \in (p + U_n) \cap W$, then, by the triangle inequality, the q_n form a Cauchy sequence in W and thus converge to a point $q \in W$. As they also converge to p by exhaustion of the neighborhood system of p and as V is Hausdorff, we get $p \in W$. □

Theorem 1.24 *Let V be a finite-dimensional vector space. Then V has exactly one Hausdorff topology, and with this topology, V is linearly homeomorphic to $\mathbb{R}^n$, which is Banach.*

Proof. By induction. Assume it holds for an $n \in \mathbb{N}$. Then on an $(n+1)$-dimensional vector space choose a basis v_i and induce a norm n from K^{n+1}, this is Banach by Theorem 1.21. Now let T be a Hausdorff topology on V. Then define $v_k^* : \sum \lambda_i v_i \mapsto \lambda_k$. As by induction and Theorem 1.23 we have that $ker v_k^*$ is closed, Theorem 1.5 implies that all v_k^* are T-continuous. On the other hand, they are n-continuous by definition. Thus $\mathbf{1} = \sum v_i^* \cdot v_i$ is continuous in both directions. □

Now let us come to the notion of *basis*:

Definition 1.25 *Let V be a vector space. An* **algebraic basis** *or* **Hamel basis** *of V is a subset B of V such that for every element v of V there is a unique* finite *subset $A = ((k_1, v_1), ...(k_n, v_n))$ of $K \times B$ with $v = \sum_{i=1}^{n} k_i \cdot v_i$.*

Let V be a Hausdorff topological vector space. A **topological basis** *of V is a subset B of V such that for every element v of V there is a unique* countable *subset A of $K \times B$ with $v = \sum_{i \in \mathbb{N}} k_i \cdot v_i := \lim_{n\to\infty} \sum_{i=1}^{n} k_i \cdot v_i$ for some counting $i \mapsto (k_i, v_i)$ of A. If the coefficient functions are continuous, then B is called* **continuous basis***, if moreover B is countable, it is called* **Schauder basis***.*

Remark: A usual additional requirement on a topological basis is countability. We will not require this here, as there are tvs whose only topological basises are uncountable, e.g. $B_c(\mathbb{N}, \mathbb{R}^n)$ for $n \geq 2$ (which in this case can be chosen to be continuous). *If* the topological basis is countable then one can choose a fixed counting and speak of a unique sequence of coefficients for every vector. Instead of countability one could require also that the basis be discrete, i.e. that for every element v of the basis we can find an open neighborhood U_v such that $U_v \cap U_w = \emptyset$ for $v \neq w$. This is automatically the case for countable basises **(exercise (23))**. A discrete Hamel basis is a Schauder basis. While the existence of a Hamel basis is always asserted by the Axiom of Choice, the same question for topological basises is, to our knowledge, still unanswered.

Theorem 1.26 *In a Hausdorff tvs, if a series of vectors converges, the series of every permutation either converges to the same vector or does not converge at all.*

Proof. Assume that $\sum x_n \to v$ and $\sum x_{\sigma(n)} \to w \neq v$. Now choose two disjoint neighborhoods U of v and W of w. Now there is a finite $F_0 \in \mathbb{N}$ such that for every finite $F \supset F_0$, we have $\sum_{n \in F} x_n \in U$. Now we take an N_0 with $F_0 \subset \{\sigma(1), ...\sigma(N_0)\}$, and an $N > N_0$ with $\sum_{i=1}^{N} x_{\sigma(i)}$. If we define $F := \{\sigma(1)...\sigma(N)\}$, then $F_0 \subset F$ and

$$W \ni \sum_{n=1}^{N} x_{\sigma(n)} = \sum_{n \in F} x_n \in U$$

in contradiction to the assumption that U and W are disjoint. $\square$

Exercise (24): Show that the unit vectors e_n with $e_n(i) = 0$ for $i \neq n$, $e_n(n) = 1$, form a Schauder basis of $B_0(\mathbb{N}, K)$, but not of $B(\mathbb{N}, K)$!
Exercise (25): Show that finite-dimensional subsets of a Hausdorff tvs are nowhere dense. Conclude with Theorem 1.20 that any Hamel basis in a complete metrizable vector space is uncountable.
Remark: In the previous exercise, completeness is necessary as $B_c(\mathbb{N}, K)$ has a countable Hamel basis consisting of the vectors e_i above.
Exercise (26): A topological space is called **separable** if it contains a countable dense subset. Show that if a tvs V has a countable Schauder basis that then it is separable!
Remark. The converse is not true: There are separable tvs without a Schauder basis as we will see later.

Theorem 1.27 (Hahn-Banach Theorem) *Let V be a real vector space and W a linear subspace. Let $p : V \to \mathbb{R}$ be* **sublinear** *(that is, $p(v + w) \leq p(v) + p(w)$ for all $v, w \in V$, and $p(\lambda v) \leq \lambda p(v)$ for all $\lambda \in \mathbb{R}, v \in V$) and $f \in L(W, \mathbb{R})$ with $f(w) \leq p(w)$ for all $w \in W$. Then there is an $F \in L(V, \mathbb{R})$ with $F|_W = f$ and $F(v) \leq p(v)$ for all $v \in V$, that is, f can be extended to a linear functional on all of V still dominated by p.*

Proof. The proof is based on the Axiom of Choice in the form of Zorn's Lemma and the following Lemma which allows a gradual extension of the subspace by one dimension:

Lemma 1.28 *Let $Y \subsetneq V$ be a proper subspace of V. Then there is another subspace Y' of V, $Y \subsetneq Y'$, and a linear extension f' of f on Y' such that still $f' \leq p$.*

Proof. Fix an $x \in V \setminus Y$ and define $Y' := Y + \mathbb{R}x$. Then every $y' \in Y'$ has a unique decomposition $y' = y + tx'$ where $y \in Y$ and $t \in \mathbb{R}$. Now define, for every real number a, the linear functional $f'_a(y + tx) := f(y) + a \cdot t$ on Y'. Obviously it restricts to f on Y. We want to show that for a small enough it is dominated by p. Observe that for $y_1, y_2 \in Y$ we have

$$f(y_1) + f(y_2) = f(y_1 + y_2) \leq p(y_1 - x + y_2 + x) \leq p(y_1 - x) + p(y_2 + x),$$

thus $f(y_1) - p(y_1 - x) \leq p(y_2 + x) - f(y_2)$, and

$$A := \sup\{f(y) - p(y - x) | y \in Y\} \leq \inf\{p(y + x) - f(y) | y \in Y\} =: B,$$

and we can choose a real number r with $A \leq r \leq B$ and define $f' := f'_r$. Then for any $t > 0$ we have

$$f'(y + tx) = t(f(t^{-1}y) + r) \leq tp(t^{-1}y + x) = p(y + tx),$$

$$f'(y - tx) = t(f(t^{-1}y) - r) \leq tp(t^{-1}y - x) = p(y - tx),$$

thus p still dominates f'.

Now to complete the proof of the theorem, let A be the family of all linear functionals g, whose domains are linear subspaces of V, which restrict to f on Y and which are dominated by p. They are partially ordered by restriction. Every nonempty chain (i.e. totally ordered subset) of A has an upper bound by the union of the corresponding subspaces and the definition of the linear functional by restriction. Thus by Zorn's lemma there is a maximal element F. Its domain has to be all of V as otherwise it could be enlarged as in the lemma. This concludes the proof. $\square$

This theorem is of such a general applicability that Pedersen wrote once *'It can be used every day, and twice on Sundays.'*. One of its numerous corollaries is the following:

Theorem 1.29 *Let B be a Banach space and A a linear subspace of B. Every $f \in CL(A, \mathbb{R})$ with $||f|| = C$ has an extension $F \in CL(B, \mathbb{R})$ with $||F|| = C$.*

Proof. Take $p := C \cdot || \cdot ||$. $\square$

Theorem 1.30 *Let V be a Hausdorff tvs and $C \subset V$ a closed subspace. Then V/C with the quotient topology is again a Hausdorff tvs.*

The **proof** is an **exercise (27)**.

Theorem 1.31 (by Riesz in the case of Banach spaces) *Let V be a Hausdorff tvs. V is locally compact if and only if it is finite-dimensional.*

Proof. One direction is trivial by the Heine-Borel Theorem. For the other direction assume that there is a compact neighborhood U of $0 \in V$. Let $\frac{1}{2}U := \{\frac{1}{2}u | u \in U\}$. For every $x \in U$ define $V(x) := x + \frac{1}{2}U$ which is a neighborhood of x. By compactness of U there are $x_1, ...x_n \in U$ with $U \subset \bigcup_{i=1}^n V(x_i)$. Put $M := span(x_1, ...x_n)$. We want to show that $M = V$. First observe that M is closed in V as any finite dimensional linear subspace of a Hausdorff tvs is (first note that it is Hausdorff and then use Tychonoff's Theorem and Theorem 1.23). Thus the quotient V/M is a Hausdorff tvs by Theorem 1.30, and the projection $\pi : V \to V/M$ is continuous and open as always in the quotient topology, so $W := \pi(U)$ is a compact neighborhood of $0 \in V/M$. By construction, $U \subset M + \frac{1}{2}U$. Thus using that π is linear and vanishes on M, we have $W \subset \frac{1}{2}W$ and by induction $2^j W \subset W$ for all $j \in \mathbb{N}$, that means $W = V/M$, so V/M is compact. So it cannot contain any one-dimensional closed subspaces homeomorphic to K as the latter one is not compact which leaves only the case $V/M = \{0\}$. □

Let us from now on restrict ourselves to **locally convex Hausdorff tvs** or **lhs** for short, for which we require that they be Hausdorff and that every neighborhood of a point v contain a convex neighborhood of v.

Exercise (28). Show that all tvs considered so far are lhs!

Theorem 1.32 (Separation theorem) *Let V be a lhs over K. Then $CL(V,K)$ separates points of V, that means, if $p, q \in V$, $p \neq q$, then there is an $f \in CL(V,K)$ with $f(p) \neq f(q)$.*

Proof. Consider first $K = \mathbb{R}$. Without restriction of generality, let $p = 0$. Then take a neighborhood U of 0 not intersecting q and an open and convex subneighborhood C_0 of U, define $C := C_0 \cap (-C_0)$ which is an open, convex and starshaped neighborhood of 0. Define $p_C : V \to \mathbb{R}$,

$$p_C(v) = \max\{\sup\{|r| : r \cdot v \in C\}, 1\}.$$

As C is convex, this is a sublinear function as the two arguments in the maximum are sublinear functions and

$$max\{a(v+w), b(v+w)\} \leq max\{a(v)+a(w), b(v)+b(w)\} \leq max\{a(v), b(v)\}+max\{a(w), b(w)\}.$$

Now define $W = \mathbb{R} \cdot q$ and $f \in L(W, \mathbb{R})$ by $f(\lambda \cdot q) = \lambda$. On W we have $f(w) \leq p(w)$. Thus by Hahn's Extension Theorem we can find an $F \in L(V, \mathbb{R})$ with $F(v) \leq p(v)$ for all $v \in V$. This implies that C is a neighborhood of 0 with $F(C)$ bounded in $\mathbb{R}$. Linearity of F implies then that F is continuous. By definition $F(q) = 1 \neq 0 = F(0)$. For the complex case use the fact that one can write a general complex linear functional A as $Av = Bv + iCiv$ for two real linear functionals B and C. □

Definition 1.33 *Let V, W be tvs, let $U \subset V$ be open, let $p \in U$. A map $f : U \to W$ is called* **differentiable at** *p iff there is a linear map $A_p : V \to W$ with*

$$A_p(v) = \lim_{t \to 0} \frac{f(p + t \cdot v) - f(p)}{t}$$

for all $v \in V$.

This is well-defined as the argument of the limit is defined for t in an interval around 0 because scalar multiplication is continuous. Trivially A_p as above is unique if it exists.

Definition 1.34 *Let V, W be topological vector spaces, let $U \subset V$ be open. A map $f : U \to W$ is called* **differentiable** *iff it is differentiable at every $p \in U$ and if the map $f' : U \times V \to W$, $f'(p, v) := A_p(v)$, is continuous w.r.t. the product topology. In this case f' is called* **the first derivative of f**. *If it exists, the* **(n+1)th derivative** *$f^{(n+1)} : U \times V^{n+1}$, is defined by*
$f^{(n+1)}(p, v_1, ...v_n) = \frac{d}{dt}|_{t=0}(f^{(n)}(p + tv_{n+1}, v_1, ...v_n))$,
and in this case the map is called **(n+1) times differentiable** *or a C^{n+1}* **map**. *We set $C^\infty(U, W) := \bigcup_{i=1}^{\infty}(U, W)$ and call this the space of* **smooth maps**. *The* **k-th differential** *$d^k f(p) \in L(E^k \to F)$* **of** *$f$* **at** *$p$ is defined as the multilinear part $d^k f(p)(e_1, ...e_k) = f^{(k)}(p, e_1, ...e_k)$.*

Theorem 1.35 *Let $a \in K$, $U \in V$ open and $f, g : U \to W$ be C^n maps. Then $f + ag$ is a C^n map and*

$$(f + ag)^{(n)} = f^{(n)} + ag^{(n)},$$

in other words, the C^m maps from U to W form a K vector space $C^m(U, W)$ and the map $^{(n)} : C^m(U, W) \to C^{m-n}(U, W)$ is linear.

Proof. The pointwise differentiability and the form of the derivative is trivial. For continuity, observe that for a number a and two continuous maps $F, G : \tilde{U} \to W$ also $F + aG$ is continuous. □

Corollary 1.36 *Let V, W, X be topological vector spaces, $U \in V$ open and $f : U \to W, g : U \to X$ be C^n maps. Then $h = (f, g) : U \to W \oplus X$ is a C^n map and $h^{(n)} = (f^{(n)}, g^{(n)})$.*

Theorem 1.37 (chain rule and pointwise chain rule) *Let V, W, X be tvs, $U_1 \subset V$ and $U_2 \subset W$ be open, let $g : U_1 \to U_2$, $f : U_2 \to X$ be continuous in their domains of definition and differentiable at $p \in U_1$ resp. $g(p)$. Then $f \circ g$ is differentiable at p and*

$$(f \circ g)'(p, v) = f'(g(v), g'(p, v)).$$

If f and g are differentiable in their respective domains of definition, so is $f \circ g$.

Proof. We compute as usual

$$\begin{aligned}
f'(g(v), g'(p, v)) &= \lim_{s \to 0} \frac{f(g(p) + sg'(p, v)) - f(g(p))}{s} \\
&= \lim_{s,t \to 0} \frac{f(g(p) + s\frac{g(p+tv)-g(p)}{t}) - f(g(p))}{s} \\
&= \lim_{t \to 0} \frac{f(g(p) + t\frac{g(p+tv)-g(p)}{t}) - f(g(p))}{t} \\
&= \lim_{t \to 0} \frac{f(g(p + tv)) - f(g(p))}{t} \\
&= (f \circ g)'(p, v).
\end{aligned}$$

Continuity follows as above (as preserved by composition). □

The proof of the following theorem is a straightforward **exercise (29)**:

Theorem 1.38 *Let V, W be tvs, let $A = A(0) + L$ be an continuous affine map from V to W. Then A is differentiable and $A'(p, v) = L(v)$.* □

Theorem 1.39 (Euler's Theorem) *Let E, F be lhs. $U \subset E$ open and $f \in C^r(U, F)$. Then $f^{(k)}$ is symmetric in its last k arguments, i.e. $d^k f(u)$ is a linear totally symmetric map for all $u \in U$.*

Proof. We first focus on the first nontrivial case $k = 2$. Thus let $u \in U$, $v, w \in E$ be given, and we want to show $d^2 f(u)(v, w) = d^2 f(u)(w, v)$. Consider the affine map $A : \mathbb{R}^2 \to E$, $A(a, b) := u + av + bw$. Now it is easy to see **(exercise (30)!)** that $d^2 f(u)(v, w) = d(f \circ A)(0)(e_1, e_2)$. Thus our task reduces to the one to show that for $B := f \circ A : V := A^{-1}(U) \to F$ we have $dB'(0)(e_1, e_2) = dB'(0)(e_2, e_1)$. Now let an arbitrary $L \in CL(F, \mathbb{R})$ be given, then for $C := L \circ B$ and because of the Separation Theorem, it is enough to prove that $C : A^{-1}(U) \to \mathbb{R}^2$ has symmetric second derivatives at 0 which is a classical fact (proven with the finite-dimensional mean value theorem).
Now for $k > 2$, we proceed inductively: if $d^k f(u)$ is totally symmetric for all u, $d^{k+1} f(u)$ has to be symmetric in the last k entries. On the other hand,

$$d^{k+1} f(u)(v_1, ...v_{k+1}) = d^2 \Phi(u)(v_1, v_2),$$

where $\Phi(u) = d^{k-1} f(u)(v_3, ...v_k)$, thus it is symmetric in the first two entries. As the symmetric group S_k is generated by the permutation of the first two elements and the set of all permutations of the last $k - 1$ elements, we are done. □

Now we want to define the integral.

Let E be an lhs. Consider $C^0([a, b], E)$, the space of all continuous maps from $[a, b]$ to E topologized by the compact-open topology. This is a tvs **(even an lhs: exercise (31)!)**. A function $f \in C^0([a, b], E)$ is called **piecewise linear** iff there is a partition $a = t(0) < t(1) < ... < t(n) = b$ of $[a, b]$ such that $f|_{[t(i), t(i+1)]}$ is linear for all $i = 0, ...k - 1$. The piecewise linear maps form a subspace $PL([a, b], E)$ of $C^0([a, b], E)$ which is dense: Given a map $c \in C^0([a, b], E)$ and a neighborhood A of c, then we can find a neighborhood $([a, b], O) \subset A$ for an open set $O \subset E$. Now for every point p of $[a, b]$ find a convex neighborhood $U_p \subset O$ of $c(p)$, then the U_p form an open covering of $c([a, b])$, and any finite subcovering gives rise to a piecewise linear map contained in (C, O).

For $c \in PL([a, b], E)$ define

$$I_{ab}(f) := \int_a^b f(t)dt := \sum_{i=1}^{n} \frac{1}{2}(f(t(i)) - f(t(i-1)))(t(i) - t(i-1)),$$

then $I_{ab} \in CL(PL([a, b], E), E)$ and extends continuously on all of $C^0([a, b], E)$.

Theorem 1.40 *The integral has the following properties:*

1. *$I_{ab} : C^0([a, b], E) \to E$ is linear and continuous,*
2. *For all $l \in CL(E, \mathbb{R})$ we have $l(\int_a^b f(t)dt) = \int_a^b l(f(t))dt$,*
3. *For all continuous seminorms $|| \cdot ||$ on E we have $|| \int_a^b f(t)dt|| \leq \int ||f(t)||dt$,*
4. *$\int_a^b f(t)dt + \int_b^c f(t)dt = \int_a^c f(t)dt$.*

Proof by proof for piecewise linear maps and continuous extension □

For a C^1 curve c in a lhs E we define $c^{\cdot}(t) = c'(t, 1)$.

Theorem 1.41 (fundamental theorem of calculus) *Let E be a lhs. Let $f : [a,b] \to E$ be a C^1 curve, then*

$$f(b) - f(a) = \int_a^b f^{\cdot}(t).$$

If $g : [a,b] \to E$ is a C^0 curve, and if we define

$$f(t) := \int_a^t g(s)ds,$$

then f is C^1 with $f^{\cdot}(t) = g(t)$.

Proof. We will reduce this theorem to the theorem in the case $E = \mathbb{R}$. So let f be a C^1 curve, and let $A \in CL(E, \mathbb{R})$. Then $A \circ f \in C^1(\mathbb{R}, \mathbb{R})$ and by the chain rule we have $(A \circ f)^{\cdot}(t) = A \circ f^{\cdot}$. Then the fundamental theorem of calculus for real functions tells us that

$$A(f(b)) - A(f(a)) = A(\int_a^b f^{\cdot}(t)dt).$$

The rest is an application of the Separation Theorem.
For the second part, observe that

Lemma 1.42 *For $g : [a,b] \to E$ is a C^0 curve we have*

$$\int_t^{t+h} f(s)ds = h \int_0^1 f(t + hr)dr.$$

Proof of the lemma. In the case $E = \mathbb{K}$ this is known (substitution $s = t + hr$). In the general case it can be proved by composing with arbitrary continuous linear functionals and applying the Separation Theorem.

By the lemma we have

$$\tilde{f}(t,h) := \frac{f(t+h) - f(t)}{h} = \int_0^1 f(t + hr)dr.$$

Now $\tilde{f}$ is continuous (consider the right-hand side and take into account that the integral is continuous and as pointwise scalar multiplication and vector addition is continuous), thus f is differentiable with $f' = g$. $\square$

Definition 1.43 *Let V and W be tvs. Then the* **tensor product** *of V and W is the linear subspace of the tvs $C^0(V \times W, K)$ (topologized with the compact-open topology) which consists of the maps linear in both arguments, i.e. the maps f with $f(\lambda v_1 + v_2, w) = \lambda f(v_1, w) + f(v_2, w)$ and similar in the second argument. It is denoted by $V \otimes W$.*

Exercise (32): Find a basis of $V \otimes W$ for V, W finite-dimensional!